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Intermittency from Maximum Entropy Distribution

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A maximum entropy distribution has been formulated in which the imposed constraint contains a stochastic (rather than a deterministic) variable. The distribution depends on the observational bin size through the smoothing of population by intrabin averaging. Moments of fluctuations calculated with this distribution give bin-size dependences (intermittency exponents) that agree reasonably with those obtained from the size dependence in nuclear multifragmentation. The exponents depend on the spread of the stochastic mechanism (supposed to be a cascading, multiplicative process) and on the magnitude of the constraint imposed. An information-theoretic interpretation is provided for the relation between statistical and mechanism-induced (dynamic) fluctuations.

KEY WORDS: Maximum entropy; fragment distribution; nuclear multifragmentation; intermittency; fluctuation.

1. INTRODUCTION

The existence of sharp fluctuations superimposed on a regular spectral pattern, called "intermittency," arose in turbulence[1-3] and was predicted to occur elsewhere[4] mainly due to random processes operating in a multiplicative (not additive) fashion. A Landau-inspired idea (called Kolmogorov's third hypothesis) envisaged the cascade of energy excess from large to small size scales in turbulent fluids.

More recently intermittency was found and theoretically described in nuclear multifragmentation when the products were analyzed as distributed in the rapidity scale.[5] A cascade process was again postulated, which is indeed not foreign in this context.[6,7] Similar statistical effects have been noted in e^+–e^- annihilation data[8-11] and in hadronic interactions.[12] There were, however, two new aspects not encountered in turbulence: One

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was the surmise of a phase transition of a nonthermal kind[13,14] (also met with in spin-glasses[15,16]), which had its conceptual roots in the more particular, specialized description of nuclear fragmentation as a percolation phenomenon[17] (see ref. 16 for the relation between these kinds of phase changes and of diffusion-limited aggregation process, as well as for a free energy interpretation in a cascade model). Second, in nuclear multifragmentation the scale change was due to the successive refinement (of the rapidity measurements) or, in the current parlance, the decrease of observational bin sizes. The repercussion of changes in the observational method on the physical process is not a trivial matter and has not been fully accounted for in the theories proposed. In this context one is reminded first of recent philosophical speculations (not a derogatory word) that connect the limits of informational capacity to the size or age of the universe and second, on a more earthly plane, of the dependence of fragment size predictions by maximum entropy methods.[18] In the present work it is shown how the choice of bin sizes influences the averaging of data and thereby affects the observed fluctuations.

A moment analysis of fragment-size fluctuations has recently indicated intermittency behavior in the breakup of high-energy nuclear matter studied in emulsion and fragment-size distributions obtained in a selective manner.[19] Some of these data are treated in the present paper. This is done here by the adaptation of the maximum entropy method (MEM) to include a random process, in a manner not heretofore proposed, although MEM has been used for chaos,[20] for nuclear fragmentation mechanisms,[21,22] and for transport phenomena occurring on a mesocopic scale.[23] Our approach gives in a general and simple manner MEM distributions when randomness is present in a cascading process of nuclear breakup.

This type of inherent randomness is called "dynamic fluctuations" and has been distinguished from statistical fluctuations.[5,24] The advocated use of factorial moments instead of ordinary moments has separated in an extraordinarily elegant way the two types of irregularities.[5] We believe that an information-theoretic distinction between the two types related to noisy channels[25,26] has something to offer by way of clarification and this is given in the next section, although here no new results are reached.

2. TWO TYPES OF FLUCTUATIONS

Following Bialas and Peschanski[5,13] and other authors,[20,21] one can make a distinction between so-called dynamic and statistical fluctuations. Experimental scatter includes both and it is the task of theory to disentangle them. In this section we show how the information-entropy description reproduces the known results and provides a clear understanding of the two types of fluctuations in terms of familiar concepts, like prior and conditional (posterior) probabilities. The theory (if it can be called that) of this section will then enable us to continue the treatment to the prior probabilities only, which represent *dynamic* scatter, and to forget about statistical fluctuations whose role in the theory is dealt with here.

We consider the outcomes n'_m of measurements, such that n'_m is the number of observed occurrences of event-type (bin) labeled m $(=1,\ldots, M)$, summing to a fixed number N,

$$\sum_{m=1}^{M} n'_m = N \tag{1}$$

We suppose that there is an underlying distribution

$$P(\{n_m\}) \tag{2}$$

for the frequency of events that occur; however, not all events are obsrved or not in the proportion expected from (2). We require the distribution of observed events, namely

$$P'(\{n'_m\})$$

Whether the pursued analogy is that of noisy channels or conditional distributions or priors–posteriors, the following relation is evident:

$$P'(\{n'_m\}) = \sum_{\{n_m\}} P(\{n'_m\} \mid \{n_m\})\, P(\{n_m\})$$

As a matter of personal preference, we shall regard $P(n_m)$ as priors, the $P'(n'_m)$ as observed distributions, and $P(n'_m \mid n_m)$ as conditionals. The last induces statistical fluctuations in P', which however, also contains dynamic fluctuations arising from $P(n_m)$ (as long as this is not a delta function). We shall treat "the prior" P in the next section; here we derive $P(n'_m \mid n_m)$ from maximum-entropy considerations.

Associating probabilities p_m and p'_m with observed numbers according to

$$\begin{aligned} p_m &= n_m/N \\ p'_m &= n'_m/N \end{aligned} \tag{3}$$

we understand by p'_m (p_m) the probability of each one of the n'_m (n_m) events taking place. By (1), there are N events in all; we can therefore write the unconditional entropy $S\{p'_m\}$ for all these events in the following form:

$$S\{p'_m\} = N \sum_{\{p_m\}} P(\{p'_m\} \mid \{p_m\})\, S(\{p'_m\} \mid \{p_m\}) - \sum_{\{p_m\}} P(\{p'_m\} \mid \{p_m\}) \log P(\{p'_m\} \mid \{p_m\}) \tag{4}$$

where the first term includes a sum over all combinations of observations of a single event (that eventually produces $n_4 = 1$, say, and $n_m = 0$, $m \neq 4$) and involves partial entropies, which are, in fact, the cross-entropies for events $\{p'_m\}$ following the hypothesis $\{p_m\}$ for the event. Since there are N events with the same *a priori* cross-entropy, the information deficiency becomes multiplied by N. The last term represents the information deficiency for the possible choices of the conditional probabilities

$$P(\{p'_m\} \mid \{p_m\})$$

Maximize $S\{p'_m\}$ by solving

$$\frac{\partial S}{\partial P(\{p'_m\} \mid \{p_m\})} = 0$$

and one obtains

$$P(\{p'_m\} \mid \{p_m\}) = e^{NS(\{p'_m\} \mid \{p_m\}) - \mu}$$

where μ is a normalizing constant for the probabilities. However, the conditional entropy is known in the form

$$S(\{p'_m\} \mid \{p_m\}) = -\sum_m p'_m \log(p'_m/p_m)$$

Introducing the numbers n instead of probabilities p, through (3), and normalizing, we derive

$$P(\{p'_m\} \mid \{p_m\}) = N! \prod_m \frac{n_m^{n'_m}}{n'_m!} \tag{5}$$

or the multinomial or Bernoulli distribution. The remarkable relationship between experimental-factorial distributions and the ordinary moment of the underlying distribution [$P(n_m)$ in our nomenclature] that has been described by Bialas and Peschanski is based on the form of the conditional given in (5).

Our work in the sequel relates to the scatter present in the (prior or underlying) $P(n_m)$ distribution. The present section makes clear the entropy interpretation of the conditional.

3. THE PHYSICAL MODEL AND ITS TREATMENT

Our description aims at establishing a clear relation between the *physical processes* taking place in the ensemble and the way *measurements* are taken. At the root of the physical processes are the N basic states of the system (label j), which could be identified with the eigenstates of some Hamiltonian, but, alternatively, could represent (in a fragmentation process) the sizes of the fragment clusters, disregarding internal eigennumbers of fragments. We assume that the physical process distributes the initial ensemble into the various states, and does so by a cascade process of q_f (f for final) steps, at the end of which the ensemble is distributed among the N states, in a manner similar to the energy distribution in turbulence.

If the cascade steps operate on g equal alternatives, then the total number of states in q_f steps is

$$N = g^{q_f} \tag{6}$$

irrespective of the way measurements or bin sizes are chosen. We now assume that with each of the final states j we can associate a "strength factor" ϕ_j that depends on the history of the cascade process. In the spirit of the maximum entropy method we shall impose an additive constraint on ϕ_j (though we might have used the strength factors in a rate-equation approach, with no less justification).

The constraint involves $\bar{n}_j$, the mean number of systems belonging to state j, in the form

$$NF = \sum_{j=1}^{N} \phi_j \bar{n}_j \tag{7}$$

where F is a constant that depends on the nature of the process (including initial conditions, ensemble preparation, and possibly the mode of cascading). It would be natural to call NF the totality of a conserved quantity (e.g., energy), but in MEM, NF will also depend on our knowledge, which is subjective. If j is ordered, e.g., with the size of the fragments, ϕ_j will be a function of j exhibiting some rough regularity. We shall be concerned almost exclusively with the *deviations* in ϕ_j from this regularity (the fluctuating part) and can therefore take the regular part of ϕ_j as flat. For the fluctuations we shall assume, analogously to Kolmogorov's third hypothesis,[2,3] that they accumulate multiplicatively at each cascade step, in a manner that has been described before.[1–5] In fact, measurements made on individual j-states will find $\log \phi_j$ distributed (approximately) normally around its mean and with a deviation of $q_f \sigma^2$, σ^2 being the deviation at each cascade step. (Other distributions than the normal are conceivable and will not in general lead to a variance proportional to q_f, the length of the cascade. If the strength factor ϕ is reached by a Brownian-type or Levy–Wiener–Bachelier process, its distribution will have a variance proportional to q_f, but the intermittency exponents will not depend on the bin size in the experimentally observed fashion.) If, however, one scans r (number of) bins simultaneously, or equivalently, if one reduces the bin number from N to $N/r = M$, then the fluctuation is each bin is reduced, approximately to $q\sigma^2$, where

$$M = g^q, \qquad r = g^{q_f - q}, \qquad q < q_f \tag{8}$$

The reason for the reduction is the averaging over r strengths in each bin.

Let us describe the scanning procedure in the MEM formalism, namely by the introduction of priors (used in the sense of rescaling of observations, rather than is the sense of some background belief as in the previous section). We write the Langrangian [a term used for the information entropy together with the constraint, Eq. (7)] in the original, state representation

$$\begin{aligned} L = &-\sum_{j, n_j} p(n_j, j) \log p(n_j, j) + \lambda \left[\sum_{n_j, j} \phi_j p(n_j, j)\, n_j - NF \right] \\ &+ \sum_j \mu_j \left[\sum_{n_j} p(n_j, j) - 1 \right] \end{aligned} \tag{9}$$

where $p(n_j, j)$ is the probability to find n_j occurrences of the j-state. The corresponding probability $P(n_m, m)$ in the bin representation (with m labeling the bins from 1 to M) is obtained with a prior $\Pi(m)$, such that

$$P(n_m, m) = \Pi(m)\, P[n_j, j(m)]$$

The prior $\Pi(m)$ is simply the Jacobian for passing from the (j) to the bin (m) representation, namely

$$\Pi(m) = \frac{dj(m)}{dm} = r \tag{10}$$

The first term in (9) becomes (noting the changed summation)

$$S(\{m\}) = -\sum_{m, n_m} P(n_m, m) \log[P(n_m, m)/\Pi(m)]$$

The second sum appearing in (9) is remanipulated, as follows:

$$\sum_j \phi_j p(n_j, j)\, n_j = \sum_{m, n_j} \Pi(m)\, \phi_{j(m)}\, p(n_j, j(m))\, n_j \tag{11}$$

$$= \sum_{m, n_m} \frac{\sum_{i \in m} \phi_i}{\sum_{i \in m} 1}\, p(n_m, m)\, n_m \tag{12}$$

$$= \sum_{m, n_m} \bar{\phi}_m P_p(n_m, m)\, n_m \tag{13}$$

where, in passing from (11) to (12), we "decoupled" (in the sense of a mean field approximation) the summations over j in each bin (represented by $\sum_{j \in m}$) and in (13) we have denoted the bin averaged strengths by $\bar{\phi}_m$.

Substituting in (9), we obtain from the maximum entropy condition, namely $\partial L/\partial P(n_m, m) = 0$, the solutions

$$P(n_m, m) = re^{-\lambda \bar{\phi}_m n_m - \mu_m} = r(1 - e^{-\lambda \bar{\phi}_m})\, e^{-\lambda \bar{\phi}_m n_m} \tag{14}$$

The mean number of states in the m-bin is

$$\bar{n}_m = r(e^{\lambda \bar{\phi}_m} - 1)^{-1} \tag{15}$$

which is to be compared to the MEM solution in the j-state representation,

$$\bar{n}_j = (e^{\lambda \phi_j} - 1)^{-1} \tag{16}$$

The Lagrange multiplier in (15) is to be evaluated from the constraint in its bin averaged form, namely

$$r \sum_{m=1}^{M} \bar{\phi}_m (e^{\lambda \bar{\phi}_m} - 1)^{-1} = NF \tag{17}$$

The method used for evaluating (17) and obtaining NF will be described after introduction of the i-moments employed in intermittency-exponent analysis.

The Fluctuation i-Moments. The fluctuation in the m-bin occupation number is $n_m/\bar{n}_m$, and the overall average is $N/m = r$, so that the fluctuation i-moment can be defined as

$$C_i = r^{i-1} \frac{1}{M} \sum_{m}^{M} \sum_{n_m} \left(\frac{n_m}{\bar{n}_m}\right)^i P(n_m, m) \tag{18}$$

It turns out that the following result holds to a remarkable accuracy for the MEM solution given in (14) for i moderately large:

$$\sum n_m^i P(n_m, m) \approx K_i (\lambda \bar{\phi}_m)^{-i} \tag{19}$$

with K_i independent of $\lambda \bar{\phi}_m$.

A saddle point evaluation indeed reproduces the above result with $K_i \simeq i!$ To see this, we introduce from (14) into (19) the n_m-dependent part of $P(n_m, m)$, so that the summand in (19) is $\exp[i \log n_m - (\lambda \bar{\phi}_m) n_m]$ with a maximum at $n_m = (\lambda \bar{\phi}_m)/i$. What is remarkable is that the result (19) was confirmed by numerical computation (summation) to a high degree of accuracy and for a wide range of values of the parameter $\bar{\phi}_m$ ($< i$). We therefore proceed to the calculation of the following bin-average expression for C_i, strictly valid for i large

$$C_i \approx \frac{1}{M} \sum_{m=1}^{M} (e^{\lambda \bar{\phi}_m} - 1)^i (\lambda \bar{\phi}_m)^{-i} \tag{20}$$

Following previous works on intermittency,[4,5] and proceeding in the spirit of our remarks about the distribution of $\bar{\phi}_m$, we rewrite the last expression as a sum over the $\bar{\phi}_m$ distribution in the form

$$C_i = \frac{\lambda}{(\pi q \sigma^2)^{1/2}} \int_0^\infty d\bar{\phi}\, (\lambda \bar{\phi})^{-i-1} (e^{\lambda \bar{\phi}} - 1)^i \exp\left[\frac{-(\log \bar{\phi} - \log \bar{\phi}_0)^2}{2q\sigma^2}\right] \tag{21}$$

This method for evaluating (20) is essentially equivalent to the f–α method, also called the multifractal model,[13] but is more accurate, since it requires less reliance on asymptotic behavior.

In (21), $\bar{\phi}_0$ represents a mean value of $\bar{\phi}_m$. Unlike the deviation $q\sigma^2$, it does not depend on the bin number M or on the exponent $q = \log M / \log g$. Therefore the presently used mechanism differs from those previously proposed[5]: as will be shown, the term linear in i that appears in the intermittency exponent is due to fluctuation. The Lagrange multiplier that appears in (21) is obtained by solving (17). Employing the distribution function, we obtain an equation similar to that in (21), except that C_i is replaced by F and the two factors in parentheses under the integral are exchanged for

$$(e^{\lambda \bar{\phi}} - 1)^{-1} \tag{22}$$

Writing the intermittency exponent f_i as

$$f_i \sim \log C_i / \log M \tag{23}$$

$$= (\log C_i / \log g)\, q^{-1} \tag{24}$$

we find f_i as a function of the constraint parameter F [in (17)] and of the deviation σ^2. The branching number g enters as a scaling parameter only; the mean $\bar{\phi}_0$ is irrelevant to the results, though it allows physical interpretation to be meaningful. In practice, we have evaluated the integrals over $\bar{\phi}$ by a saddle point procedure.

Values of the moments calculated for a few values of i are shown in Fig. 1 as function of the number of bins M, for a branching number per

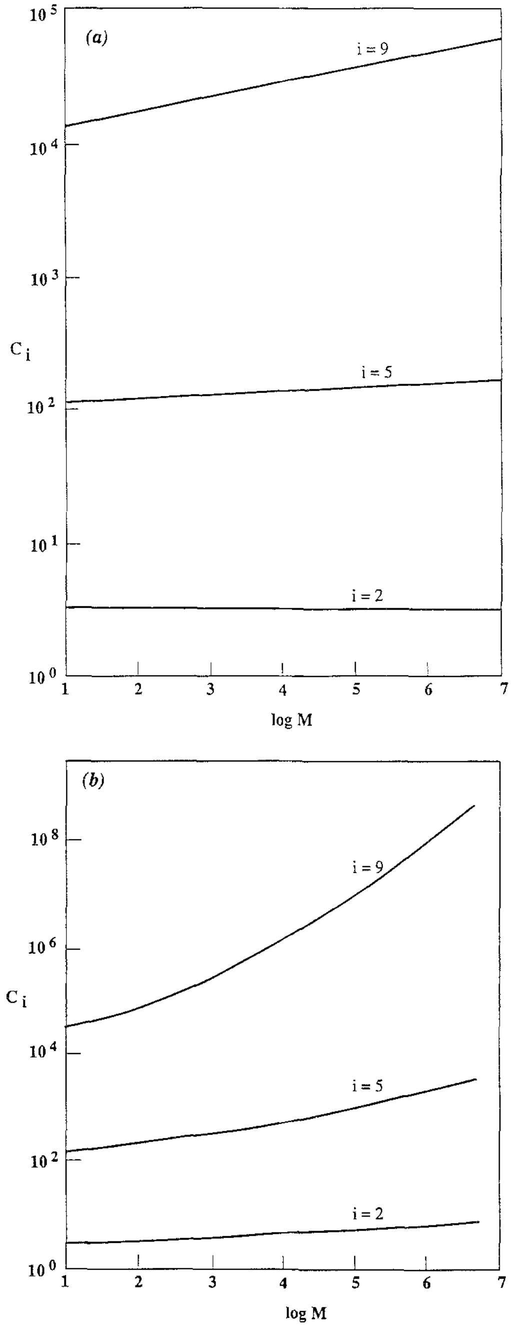

Fig. 1. Bin-number dependence of some fluctuation moments C_i. The standard deviation is $\sigma^2 = $ (a) 0.01, (b) 0.04. Other parameters: $\log F$ [in the constraint, Eq. (7)] $= 1$, g (cascade number) ~ 3.

cascade $g \sim 3$ and constraint parameter $F \sim 3$. For a low value of the deviation $\sigma^2 \sim 0.01$, the plots of $\log C_i$ appear to depend linearly on $\log M$ and have increasing slopes as the moment index i increases (Fig. 1a); however, for a higher value of $\sigma^2 \sim 0.04$, $\log C_i$ increases superlinearly with $\log M$ (Fig. 1b).

The intermittency exponents f_i [Eq. (23)] increase with i, as observed in experiments for rapidity and multifragmentation of nuclei. In fact, when written in the form

$$f_i = \alpha(i-1) + \beta[(i-1)^2 + \text{higher order terms in } i] \tag{25}$$

both α and β are small compared to unity (as observed) and increase with the deviation σ^2. This is natural, since the intermittency arises from the fluctuations that the systems feel while undergoing cascade. We reemphasize that the deviations are due to the dynamics of the process and not to statistics of samplings. In fact, the chosen statistics of sampling (namely, having bins accommodating a significant number r of states) suppresses the deviations due to averaging inside the bins: what one sees is the "remainder" from the dynamic fluctuation becoming more pronounced as the bin content (r) decreases and the bin number M ($=N/r$) increases.

The plot of f_i against i is similar to that based on observed multifragmentation results, though the quadratic coefficient β is larger in the computation (Fig. 2) than in some observations.

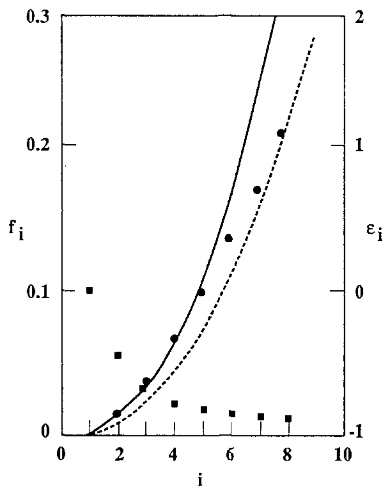

Fig. 2. Intermittency exponent f_i versus moment index i. Experimental data points (not including error bounds) are from ref. 19 for events containing several large fragments. Dots show f_i and squares (against right-hand scale) the free energy ε_i. Computed curves of f_i are for $g \simeq 3$ and for $F = 2.8$, $\sigma^2 = 0.04$ (full curve), and $F = 1.6$, $\sigma^2 = 0.01$ (broken curve).

We have investigated the dependence of the exponents on the constraint F with a view of discovering indications of some phase transition. In particular, we find for the coefficients α and β in (25) that these (Fig. 3) decrease with decreasing size of the constraint parameter F and vanish at some value of $F \ll 1$. Vanishing intermittency exponents have been observed in nuclear multifragmentation, provided that *constraints* on the *sampling procedure* are removed: e.g., by relaxing constraints on multiplicity or on finding a significant number of large fragments. It is not clear to us to what extent the *lowering* of the constraint parameter F is connected with the removal of the constraint. (We note that arguments have been given that the absence of constraints characterizes chaotic behavior.[20] It remains to be seen whether this claim is compatible with the multifragmentation result.)

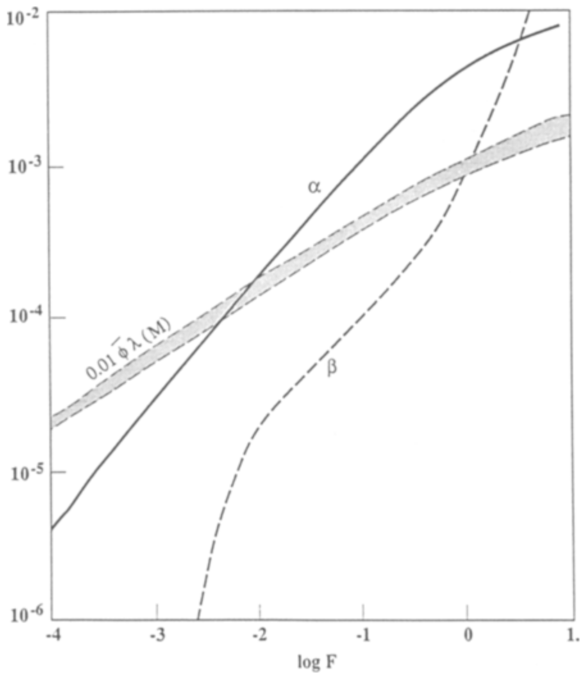

Fig. 3. Dependence of computed quantities on the constraint parameter F. The full line shows the linear coefficient α [in Eq. (20)] and the broken line the superlinear coefficient β. The shaded region represents computed values of the Lagrange multiplier multiplied by $0.01\bar{\phi}_0$ ($\bar{\phi}_0$ is the geometric mean of the interaction strengths over the cascade process). The bottom of the shaded region is for $q = \log M/\log g = 1$ (where M is the number of bins and g the branching number in the cascade process) and the top of the region is for $q = 5$. Parameters in the calculation: $\sigma^2 = 0.04$, $g \simeq 3$.

The minimum i_{crit} of the free energy

$$\varepsilon_i = (f_i + 1)/i - 1 \tag{26}$$

locates the critical index for a phase transition.[5,15] For the representation in (25), one finds

$$i_{\mathrm{crit}} = [1 + (1 - \alpha)/\beta]^{1/2} \tag{27}$$

which is well approximated for the small values of α and β in our results by

$$i_{\mathrm{crit}} \simeq (1/\beta)^{1/2} \tag{28}$$

In the same limit, the energy minimum lies at

$$\varepsilon \simeq 2\sqrt{\beta} \tag{29}$$

The experimental ε_i based on ref. 12 are plotted (squares) in Fig. 2 against the right-hand scale and indicate the presence of a minimum in ε_i (if at all) well beyond the last experimental point $i = 8$.

The dependence of the computed critical energy parameters on the constraint parameter F is readily seen from Fig. 3, using formulas (28) and (29).

4. DISCUSSION

Using physical ideas that have in the past been associated with intermittency, we have adapted the entropy maximization method to obtain a precise connection between the fluctuation distribution and a stochastic mechanism. The role of the bin size has been clarified. The calculated fluctuation moments depend, as expected, on the spread (σ) in the stochastic mechanism and on the numerical value [F in Eq. (12)] of the constraint. Our calculated moments resemble quite well those observed for the fluctuation of sizes in multifragmented nuclei.[19] This appears further to justify the use of the maximum entropy method in fragmentation.[27] However, we have not found support from our results for a cooperative effect (phase transition) in multifragmentation.

The availability of distributions should encourage a more extensive use of higher moments of fluctuation, as function of observational bin sizes, in other fields where stochastic mechanisms operate (e.g., surface roughness, growth processes) with a view of learning about the nature of the interaction. Theoretically, further studies ought to include the influence of more discriminative processes than the flat background mechanism used by us.

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REFERENCES

1. A. M. Oboukhov, *J. Fluid. Mech.* **13**:77 (1962).
2. A. N. Kolmogorov, *J. Fluid. Mech.* **13**:82 (1962).
3. V. R. Chechetkin, V. S. Lutovinov, and A. Yu. Turgin, *J. Stat. Phys.* **61**:589 (1990).
4. Ya. B. Zel'dovich, S. A. Molchanov, A. A. Ruzmařkin, and D. D. Sokolov, *Sov. Phys. Usp.* **30**:353 (1987) [*Usp. Fiz. Nauk* **152**:3 (1987)].
5. A. Bialas and R. Peschanski, *Nucl. Phys. B* **273**:703 (1986); **308**:857 (1988); *Phys. Lett. B* **207**:59 (1987).
6. J. Cugnon, *Z. Phys. A Atomic Nuclei* **327**:187 (1987).
7. G. Monetti, Charm Fragmentation, in *International Symposium on Quark Physics, Ithaca N.Y.* (American Institute of Physics, New York, 1989), pp. 179–190.
8. P. Bozek and M. Ploszajczak, *Phys. Lett. B* **254**:502 (1991).
9. B. Buschbeck, R. Lipa, and R. Peschanski, *Phys. Lett. B* **215**:788 (1988).
10. W. Ochs and J. Wosiek, *Phys. Lett. B* **214**:617 (1988).
11. R. Szwed, G. Wrochna, and A. K. Wroblewski, *Mod. Phys. Lett.* **5**:1851 (1990).
12. C. Chin and R. Hwa, *Phys. Lett. B* **236**:466 (1990).
13. R. Peschanski, *Nucl. Phys. B* **327**:144 (1989).
14. H. Statz, *Nucl. Phys. B* **326**:613 (1989).
15. P. Brax and R. Peschanski, *Nucl. Phys. B* **34**:65 (1990).
16. P. Brax and R. Peschanski, *Nucl. Phys. B* **353**:165 (1991).
17. X. Campi, *J. Phys. A* **19**:917 (1986); *Phys. Lett. B* **208**:351 (1988); *Nucl. Phys. A* **495**:259C (1989).
18. R. Englman, N. Rivier, and Z. Jaeger, *Phil. Mag. B* **96**:751 (1987); *J. Appl. Phys.* **63**:4766 (1988).
19. M. Ploszajczak and A. Tucholski, *Phys. Rev. Lett.* **65**:1539 (1990); *Nucl. Phys. A* **523**:651 (1991).
20. Y. Alhassid and R. D. Levine, *Phys. Rev. Lett.* **57**:2879 (1986).
21. P. A. Mello, P. Pereyra, and T. Seligman, *Ann. Phys.* **161**:254 (1985); W. A. Friedman and P. A. Mello, *Ann. Phys.* **161**:276 (1985).
22. J. Aichelin and J. Huefner, *Phys. Lett.* **136B**:15 (1984); J. Aichelin, J. Huefner, and R. Ibarra, *Phys. Rev. C* **30**:107 (1984).
23. J. L. Prichard and M. Sanquer, *Physica A* **167**:66 (1990).
24. P. Lipa and B. Buschbeck, *Phys. Lett. B* **223**:465 (1988).
25. L. Diosi, *Nucl. Instr. Meth.* **140**:533 (1977).
26. R. Englman and N. Rivier, *Nucl. Phys. B (Proc. Suppl.)* **5A**:345 (1988).
27. R. Englman, *J. Phys. Condensed Matter* **3**:1019 (1991).